

ON DECOMPOSITIONS OF QUADRINOMIALS AND RELATED DIOPHANTINE EQUATIONS

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ABSTRACT. Let A, B, C, D be rational numbers such that $ABC \neq 0$, and let $n_1 > n_2 > n_3 > 0$ be positive integers. We solve the equation

$$Ax^{n_1} + Bx^{n_2} + Cx^{n_3} + D = f(g(x)),$$

in $f, g \in \mathbb{Q}[x]$. In sequel we use Bilu-Tichy method to prove finiteness of integral solutions of the equations

$$Ax^{n_1} + Bx^{n_2} + Cx^{n_3} + D = Ey^{m_1} + Fy^{m_2} + Gy^{m_3} + H,$$

where A, B, C, D, E, F, G, H are rational numbers $ABCEFG \neq 0$ and $n_1 > n_2 > n_3 > 0$, $m_1 > m_2 > m_3 > 0$, $\gcd(n_1, n_2, n_3) = \gcd(m_1, m_2, m_3) = 1$ and $n_1, m_1 \geq 9$. And the equation

$$A_1x^{n_1} + A_2x^{n_2} + \dots + A_lx^{n_l} + A_{l+1} = Ey^{m_1} + Fy^{m_2} + Gy^{m_3},$$

where $l \geq 4$ is fixed integer, $A_1, \dots, A_{l+1}, E, F, G$ are non-zero rational numbers, except for possibly A_{l+1} , $n_1 > n_2 > \dots > n_l > 0$, $m_1 > m_2 > m_3 > 0$ are positive integers such that $\gcd(n_1, n_2, \dots, n_l) = \gcd(m_1, m_2, m_3) = 1$, and $n_1 \geq 4$, $m_1 \geq 2l(l-1)$.

1. INTRODUCTION

In paper [8] Schinzel, Pintér and Péter give an inefficient criterion for the Diophantine equation of the form

$$ax^m + bx^n + c = dy^p + e^q,$$

where a, b, c, d, e rationals, $ab \neq 0 \neq de$, $m > n > 0, p > q > 0$, $\gcd(m, n) = 1$, $\gcd(p, q) = 1$, and $m, p \geq 3$ to have infinitely many integer solutions.

In the later paper Schinzel [9] dropped the assumption $\gcd(m, n) = 1$, $\gcd(p, q) = 1$ and gives a necessary and sufficient condition for such equation to have infinitely many integer solutions.

In the recent paper Kreso [5] proved the finiteness of integral solutions for the equation

$$a_1x^{n_1} + a_2x^{n_2} + \dots + a_lx^{n_l} + a_{l+1} = b_1y^{m_1} + b_2y^{m_2},$$

where $l \geq 2$ and $m_1 > m_2, n_1 > n_2 > \dots > n_l$ are fixed positive integers satisfying $\gcd(m_1, m_2) = 1$, $\gcd(n_1, n_2, \dots, n_l) = 1$, $a_1, a_2, \dots, a_l, a_{l+1}, b_1, b_2$ are non-zero rationals, except for possibly a_{l+1} . With $n_1 \geq 3, m_1 \geq 2l(l-1)$ and $(n_1, n_2) \neq (m_1, m_2)$.

All the mentioned results relies on Bilu-Tichy Theorem [1], and theorems concerning decompositions of trinomials [2] as main ingredients. No such results for the equations involving at least three non-zero coefficients at positive powers on both sides are known mainly because we have no results concerning decompositions of lacunary polynomials with more than three non-zero terms [5]. Some partial results in this direction are given in [6].

In this note we describe all possible decompositions of quadrinomials. In the sequel we use Bilu-Tichy theorem to prove the following generalizations of Schinzel and Kreso results. More precisely, we prove the following

Theorem (A). *Let $f(x) = Ax^{n_1} + Bx^{n_2} + Cx^{n_3} + D$, $g(x) = Ex^{m_1} + Fx^{m_2} + Gx^{m_3} + H$ with $f, g \in \mathbb{Q}[x]$, $n_1 > n_2 > n_3 > 0$, $m_1 > m_2 > m_3 > 0$, and $\gcd(n_1, n_2, n_3) = 1$, $\gcd(m_1, m_2, m_3) = 1$, $(m_1, m_2, m_3) \neq (n_1, n_2, n_3)$, $ABC \neq 0$, $EFG \neq 0$ and $n_1, m_1 \geq 9$. Then the equation*

$$f(x) = g(y)$$

has only finitely many integer solutions.

Theorem (B). *Let $l \geq 4$ and $n_1 > n_2 > \dots > n_l > 0$, $m_1 > m_2 > m_3 > 0$ be positive integers. Let*

$$f(x) = A_1x^{n_1} + A_2x^{n_2} + \dots + A_lx^{n_l} + A_{l+1} \quad \text{and} \quad g(x) = Ex^{m_1} + Fx^{m_2} + Gx^{m_3}$$

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be polynomials with rational coefficients such that $\gcd(n_1, n_2, \dots, n_l) = 1$, $\gcd(m_1, m_2, m_3) = 1$, $A_1 A_2 \dots A_l \neq 0$, $EFG \neq 0$ and $m_1 \geq 2l(l-1)$, $n_1 \geq 4$. Then the equation

$$f(x) = g(y)$$

has only finitely many integer solutions.

Our results are ineffective as we use Theorem of Bilu and Tichy which relies on classical theorem of Siegel on integral points.

2. DECOMPOSITIONS OF QUADRINOMIALS

In this section we describe decompositions of quadrinomials. We will use some classical lemmas. Let us recall Mason-Stothers Theorem [7, 11].

Theorem 2.1. *Let $a(t), b(t), c(t) \in K[t]$ be relatively prime polynomials over a field of characteristic zero, such that $a + b = c$, and not all of them are constant. Then*

$$\max\{\deg a, \deg b, \deg c\} \leq \deg(\text{rad}(abc)) - 1,$$

where $\text{rad}(f)$ is the product of the distinct irreducible factors of f .

Let us recall Hajós lemma [4].

Lemma 2.2. *Let K be a field of characteristic 0. If $f(x) \in K[X]$ has a root $z \neq 0$ of multiplicity n then f has at least $n + 1$ terms.*

Proof. We use induction on n . When $n = 1$ then the statement obviously holds. For $n > 1$ let us put $f(x) = x^k f_1(x)$ where $f_1(0) \neq 0$. Then z is a root of f_1' of multiplicity $n - 1$ and f_1' has exactly one term less than f . The result follows. \square

Lemma 2.3. *Let K be a field of characteristic 0. If $f(x) \in K[x]$ satisfy the equation $f(x)^2 = x^{n_1} + Ax^{n_2} + B$ for some $A, B \in K \setminus \{0\}$ and $n_1 > n_2 > 0$ then $f(x)$ is a binomial.*

Proof. Suppose otherwise, that f has at least three non-zero terms. Let us write

$$f(x) = x^{k_1} + Ux^{k_2} + \dots + Vx^{k_3} + W,$$

for some $k_1 > k_2 \geq k_3 > 0$, and $UVW \neq 0$. Then we have

$$f(x)^2 = x^{2k_1} + 2Ux^{k_1+k_2} + \dots + 2VWx^{k_3} + W^2,$$

so $f(x)^2$ has at least four non-zero terms. \square

Lemma 2.4. *Let K be an algebraically closed field of characteristic 0. Let $f, g, h \in K[x]$ be polynomials such that $f(x) = g(h(x))$ and $\deg g > 1$ then there exists $\gamma \in K$ such that*

$$\deg(\gcd(f(x) - \gamma, f'(x))) \geq \deg h.$$

Proof. Let β be a root of $g'(x)$. We define $\gamma = g(\beta)$ then $h(x) - \beta$ divides both $f'(x)$ and $f(x) - \gamma$. \square

Now we are ready to state the main theorem of this section.

Theorem 2.5. *Let K be an algebraically closed field of characteristic 0. Let $f(x) = Ax^{n_1} + Bx^{n_2} + Cx^{n_3} + D$ for some $A, B, C, D \in K$ such that $ABC \neq 0$ and $n_1 > n_2 > n_3 > 0$. Suppose that $f(x) = g(h(x))$ for some $g, h \in K[x]$ then one of the following cases holds*

- (1) $g(x) = (Ax^{\frac{n_1}{d}} + Bx^{\frac{n_2}{d}} + Cx^{\frac{n_3}{d}} + D) \circ l^{-1}$ and $h(x) = l \circ x^d$ for some linear polynomial $l \in K[x]$, and positive integer $d \mid \gcd(n_1, n_2, n_3)$,
- (2) $g(x) = l(x)$ and $h(x) = l^{-1} \circ f(x)$ for some linear $l \in K[x]$
- (3) $g(x) = (Ax^2 + D) \circ l$ and $h(x) = l^{-1} \circ (x^{\frac{n_1}{2}} + \frac{B}{2A}x^{\frac{n_3}{2}})$ where $l \in K[x]$ is some linear polynomial. Moreover the following conditions holds $2n_2 = n_1 + n_3$ and $C = \frac{B^2}{4A}$.
- (4) $g(x) = (Ax(x - c^2) + D) \circ l$ and $h(x) = l^{-1} \circ (x^{2n_3} + cx^{n_3})$ for some linear $l \in K[x]$, and non-zero c . Moreover the following conditions holds $n_1 = 4n_3, n_2 = 3n_3$

Proof. By replacing f and g by $(Ax + D)^{-1} \circ f$ and $(Ax + D)^{-1} \circ g$ we can assume that $A = 1, D = 0$. Moreover by replacing g, h by $g \circ l^{-1}$ and $l \circ h$ for suitable linear l , we can assume that g, h are monic, and $g(0) = h(0) = 0$.

Let us write

$$g(x) = x^{a_0}(x - x_1)^{a_1} \dots (x - x_k)^{a_k},$$

with $a_0, a_1, \dots, a_k \in \mathbb{N}_+$, $x_i \neq 0$ for $i = 1, 2, \dots, k$ and $x_i \neq x_j$ for $i \neq j$, $i, j = 1, 2, \dots, n$. We have

$$h(x)^{a_0}(h(x) - x_1)^{a_1} \cdots (h(x) - x_k)^{a_k} = x^{n_1} + Bx^{n_2} + Cx^{n_3}.$$

We write $h(x) = x^d h_1(x)$, where $h_1(x)$ is some monic polynomial such that $h_1(0) \neq 0$. If $h_1 \equiv 1$ then we get $h(x) = x^d$ and $g(x) = x^{n_1/d} + Bx^{n_2/d} + Cx^{n_3/d}$. This corresponds to the first case on our list.

Let us suppose that h_1 has a non-zero root ξ . Then from Lemma 2.2 we get that the multiplicity of ξ as a root of $x^{n_1} + Bx^{n_2} + Cx^{n_3}$ is less or equal than 2, and therefore $a_0 \in \{1, 2\}$. We consider these two cases separately.

Case 1: $a_0 = 1$.

If $g(x) = x$ then we get trivial decomposition i.e. the second case on our list. Suppose that $\deg g \geq 2$. We have that

$$x^{n_1} + Bx^{n_2} + Cx^{n_3} = g(x^{n_3} h_1(x)).$$

Let us prove that $h_1(x) = h_2(x^{n_3})$ for some polynomial $h_2(x)$. Suppose otherwise, let $h_1(x)$ has non-zero coefficient c_ν at power x^ν , $n_3 \nmid \nu$ and ν is the smallest integer with this properties. Let us prove that $g(h(x))$ has at least four non-zero coefficients. We have

$$g(h(x)) = x^{n_3} h_1(x) (x^{n_3} h_1(x) - x_1)^{a_1} \cdots (x^{n_3} h_1(x) - x_k)^{a_k} = \sum_{i=1}^{\deg g} C_i (x^{n_3} h_1(x))^i,$$

where $g(x) = \sum_{i=1}^{\deg g} C_i x^i$. Polynomial $h_1(x)$ is not a monomial and thus $C_{\deg g} (x^{n_3} h_1(x))^{\deg g}$ has at least two non-zero coefficients at powers which cannot cancel with $C_j (x^{n_3} h_1(x))^j$ where $j < \deg g$. The coefficient at $x^{n_3+\nu}$ in $g(h(x))$ is equal to $C_1 c_\nu \neq 0$ it can't cancel as a coefficient at the lowest power which is not divisible by n_3 . So in that case $g(h(x))$ has at least four non-zero coefficients - a contradiction. We proved that $h_1(x) = h_2(x^{n_3})$. As a consequence we get the equality

$$x^{n_1} + Bx^{n_2} + Cx^{n_3} = g(x^{n_3} h_2(x^{n_3})).$$

Therefore $n_3 | n_1, n_3 | n_2$ say $m_1 n_3 = n_1, m_2 n_3 = n_2$ and

$$(1) \quad x^{m_1} + Bx^{m_2} + Cx = g(xh_2(x)).$$

Let us write $k = m_1 - m_2$. We claim that $h_2(x) = h_3(x^k)$ for some $h_3 \in K[x]$. By comparison of coefficients in identity (1) we get that

$$h_2(x) = x^t + \frac{B}{\deg g} x^{t-k} + \text{l.o.t.},$$

for some t . Let us prove that if a coefficient D_s at x^s in $h_2(x)$ is non-zero then $s \equiv t \pmod{k}$. Suppose otherwise, let ν be the highest power at which $h_2(x)$ has coefficient D_ν which is non-zero and $\nu \not\equiv t \pmod{k}$. We have

$$x^{m_1} + Bx^{m_2} + Cx = x^{\deg g} \left(x^t + \frac{B}{\deg g} x^{t-k} + \text{l.o.t.} \right)^{\deg g} + \text{l.o.t.}$$

Let us observe that the coefficient at $x^{\deg g + (\deg g - 1)t + \nu}$ on the right hand side is equal to $D_\nu \deg g$. It cannot cancel because it is the coefficient at the highest power x^u which satisfies $u \not\equiv \deg g + t \deg g \pmod{k}$. Of course $m_2 = \deg g + (\deg g - 1)t + t - k > \deg g + (\deg g - 1)t + \nu > 1$ so we arrive at contradiction. We know that $h_2(0) \neq 0$ so we have $h_2(x) = h_3(x^k)$.

We thus proved that

$$g(xh_2(x)) = g(xh_3(x^k)) = x^{m_1} + Bx^{m_2} + C.$$

We prove that $g(x) = xg_2(x^k)$ for some polynomial $g_2(x)$. Suppose that this is not the case and put $g(x) = \sum_{i=1}^{\deg g} C_i x^i$. Let ν be the smallest integer such that $\nu \not\equiv 1 \pmod{k}$ and $C_\nu \neq 0$. We have

$$\sum_{i=1}^{\deg g} C_i (xh_3(x^k))^i = x^{m_1} + Bx^{m_2} + C.$$

The coefficient at x^ν on the left hand side is equal to $C_\nu h_3(0) \neq 0$. Thus $\nu = m_1$ or $\nu = m_2$. On the other hand

$$m_2 = m_1 - k = \deg(g(xh_3(x))) - k \geq \nu(k+1) - k = (\nu-1)k + \nu > \nu,$$

so we arrive at contradiction.

We have

$$xh_3(x^k)g_2(x^k h_3(x^k)^k) = x^{m_1} + Bx^{m_2} + Cx,$$

therefore $m_2 - 1 = kl$ and $m_1 - 1 = k(l + 1)$ for some l . In consequence we get

$$h_3(x)g_2(xh_3(x)^k) = x^{l+1} + Bx^l + C.$$

Let us prove that $k = 1$. We write $g_2(x) = \sum_{j=0}^{\deg g_2} W_j x^j$, and get

$$(W_0 h_3(x) - C) + (x h_3(x)^{k+1}) \sum_{j=1}^{\deg g_2} W_j (x h_3(x)^k)^{j-1} = x^{l+1} + Bx^l.$$

We apply Theorem 2.1 to the above equation and get

$$l + 1 \leq \deg h_3 + (l + 1 - k \deg h_3) + 1 - 1 = l + 1 - (k - 1) \deg h_3,$$

thus $(k - 1) \deg h_3 \leq 0$ and $k = 1$. We get that

$$g(xh_2(x)) = xh_2(x)g_2(xh_2(x)) = x^{l+2} + Bx^{l+1} + Cx.$$

Let us write $F(x) = g(xh_2(x))$. From Lemma 2.4 we get that there exists $\lambda \in K$ such that

$$\deg(\gcd(F(x) - \lambda, F'(x))) \geq \deg xh_2(x).$$

We apply Theorem 2.1 to the equation

$$F(x) - \lambda = (x^{l+2} + Bx^{l+1}) + (Cx - \lambda),$$

and get

$$l + 2 \leq 2 + 1 + (\deg(F(x) - \lambda) - \deg(\gcd(F(x) - \lambda, F'(x)))) - 1 \leq l + 4 - \deg(xh_2(x)),$$

therefore $\deg h_2(x) \leq 1$. Let $h_2(x) = x + c$ for some non-zero c , then we have

$$g(x(x + c)) = x^{l+2} + Bx^{l+1} + Cx.$$

The left hand side is symmetric with respect to the line $x = -\frac{c}{2}$, and thus so is right hand side.

We get

$$x^{l+2} + Bx^{l+1} + Cx = (-x - c)^{l+2} + B(-x - c)^{l+1} + C(-x - c).$$

We compute second derivative of both sides and get

$$(l + 2)(l + 1)x^l + B(l + 1)lx^{l-1} = (l + 2)(l + 1)(-x - c)^l + B(l + 1)l(-x - c)^{l-1}.$$

which is equivalent to

$$((l + 2)x + Bl)x^{l-1} = (x + c)^{l-1}((l + 2)(x + c) - Bl),$$

as $c \neq 0$ we have $l = 2$. As a consequence we get $\deg g = 2$, and $g(x) = x(x + b)$ for some non-zero b . Finally we have that the coefficient at x^2 in $g(x(x + c))$ is equal to zero which implies $b = -c^2$. Summing up: in case of $a_0 = 1$ we get the solution of $g(h(x)) = f(x)$ of the following form

$$g(x) = x^2 - c^2x, \quad h(x) = x^{n_3}(x^{n_3} + c)$$

which corresponds to the third case on our list.

Case 2: $a_0 = 2$.

We know that $a_0 d = n_3$ therefore $d = \frac{n_3}{2}$. If $g(x) = x^2$ then $h(x)^2 = x^{n_1} + Bx^{n_2} + Cx^{n_3}$, and $h_1(x)^2 = x^{n_1 - n_3} + Bx^{n_2 - n_3} + C$. From Lemma 2.3 we get that $h(x)$ is a binomial, which corresponds to the second case on our list.

Now let $g(x) \neq x^2$, so g has at least one non-zero root, and $\deg g(x) \geq 3$. Let us prove that $h_1(x) = h_2(x^{\frac{n_3}{2}})$, for some monic polynomial h_2 . Suppose that this is not the case and assume that $h_1(x)$ has a non-zero coefficient c_ν at power x^ν , $d = \frac{n_3}{2} \nmid \nu$. Choose ν as the smallest integer with this property. We prove that $g(h(x))$ has at least four non-zero coefficients. We have

$$g(h(x)) = x^{n_3} h_1(x)^2 (x^d h_1(x) - x_1)^{a_1} \cdots (x^d h_1(x) - x_k)^{a_k} = \sum_{i=2}^{\deg g} C_i (x^d h_1(x))^i,$$

where $g(x) = \sum_{i=2}^{\deg g} C_i x^i$. Polynomial $h_1(x)$ is not a monomial and thus $C_{\deg g} (x^d h_1(x))^{\deg g}$ has at

least two non-zero coefficients, at powers which cannot cancel with $C_j (x^d h_1(x))^j$ where $j < \deg g$. Moreover, the coefficient at $x^{n_3 + \nu}$ in $g(h(x))$ is equal to $2C_2 c_\nu h_1(0) \neq 0$. It cannot cancel as a

coefficient at the lowest power which is not divisible by d . So in that case $g(h(x))$ has at least four non-zero coefficients - a contradiction. We proved that $h_1(x) = h_2(x^{n_3/2})$ and thus

$$x^{n_1} + Bx^{n_2} + Cx^{n_3} = g(x^d h_2(x^d)),$$

where $d = n_3/2$. Therefore $d|n_1, d|n_2$ say $m_1 d = n_1, m_2 d = n_2$ and

$$(2) \quad x^{m_1} + Bx^{m_2} + Cx^2 = g(xh_2(x)).$$

Let us write $k = m_1 - m_2$. We claim that $h_2(x) = h_3(x^k)$. By comparison of coefficients in the equality (2) we get that

$$h_2(x) = x^t + \frac{B}{\deg g} x^{t-k} + \text{l.o.t.},$$

for some t . Let us prove that if a coefficient D_s at x^s in $h_2(x)$ is non-zero then $s \equiv t \pmod{k}$. Suppose otherwise, let ν be the highest power at which $h_2(x)$ has coefficient D_ν which is non-zero and $\nu \not\equiv t \pmod{k}$. We have

$$x^{m_1} + Bx^{m_2} + Cx^2 = x^{\deg g} \left(x^t + \frac{B}{\deg g} x^{t-k} + \text{l.o.t.} \right)^{\deg g} + \text{l.o.t.}.$$

Let us observe that the coefficient at $x^{\deg g + (\deg g - 1)t + \nu}$ on the right hand side is equal to $D_\nu \deg g$. It cannot cancel because it is the coefficient at the highest power x^u which satisfies $u \neq \deg g + t \deg g \pmod{k}$. Of course $m_2 = \deg g + (\deg g - 1)t + t - k > \deg g + (\deg g - 1)t + \nu > 2$ so we arrive at contradiction. We know that $h_2(0) \neq 0$ and thus $h_2(x) = h_3(x^k)$.

We can write

$$x^2 h_3(x^k)^2 | g(xh_2(x)) = x^{m_1} + Bx^{m_2} + Cx^2,$$

therefore

$$\begin{aligned} h_3(x^k) | \frac{d}{dx} (x^{m_1-2} + Bx^{m_2-2} + C) \\ = (m_1 - 2)x^{m_1-3} + (m_2 - 2)Bx^{m_2-3} \\ = x^{m_2-3}((m_1 - 2)x^k + (m_2 - 2)B). \end{aligned}$$

We know that $h_3(0) \neq 0$ so $h_3(x^k) | ((m_1 - 2)x^k + (m_2 - 2)B)$, in particular $\deg h_3 \leq 1$. If $h_3(x)$ is constant then so is $h(x)$ and we arrive at contradiction.

Suppose that $h_3(x)$ is linear then $h_2(x) = x^k + c$ for some non-zero c . Let us write $g(x) = x^2 g_2(x)$. We then have

$$x^{m_1} + Bx^{m_2} + Cx^2 = x^2(x^k + c)^2 g_2(x(x^k + c)),$$

and thus

$$x^{m_1-2} + Bx^{m_2-2} + C = (x^k + c)^2 g_2(x(x^k + c)),$$

Let us prove that $g_2(x) = g_3(x^k)$ for some $g_3(x)$. Suppose that this is not the case. Let x^u be the lowest power such that g_2 has non-zero coefficient at x^u and $u \not\equiv 0 \pmod{k}$ then we have that the coefficient at x^u in $(x^k + c)^2 g_2(x(x^k + c))$ is non-zero therefore $u = m_2 - 2$ or $u = m_1 - 2$ in both cases we have

$$m_2 - 2 = (m_1 - 2) - k = (2k + \deg g_2 \cdot (k + 1)) - k \geq k + u(k + 1) \geq k + u \geq k + (m_2 - 2) > m_2 - 2.$$

So we have

$$x^{m_1-2} + Bx^{m_2-2} + C = (x^k + c)^2 g_3(x^k(x^k + c)^k),$$

in particular $k|m_1 - 2$ and $k|m_2 - 2$, say $k(s + 1) = m_1 - 2$. We have

$$(3) \quad x^{s+1} + Bx^s + C = (x + c)^2 g_3(x(x + c)^k).$$

Let us put $g_3(x) = \sum_{j=0}^{\deg g_3} W_j x^j$, and write

$$x^{s+1} + Bx^s = (W_0(x + c)^2 - C) + x(x + c)^{k+2} \left(\sum_{j=1}^{\deg g_3} W_j (x(x + c)^k)^{j-1} \right).$$

We apply Theorem 2.1 to the above equation and get

$$s + 1 \leq (2 + (s + 1 - (k + 2)) + 2) - 1,$$

so $k = 1$. We plug this information into equation (3) and get

$$(4) \quad x^{s+1} + Bx^s + C = (x + c)^2 g_2(x(x + c)).$$

In consequence

$$x^{s+3} + Bx^{s+2} + Cx^2 = (x + c)^2 x^2 g_2(x(x + c)),$$

the left hand side is symmetric with respect to the line $x = -\frac{c}{2}$, and thus so is right hand side

$$x^{s+3} + Bx^{s+2} + Cx^2 = (-x - c)^{s+3} + B(-x - c)^{s+2} + C(-x - c)^2.$$

We compute second derivative of both sides and get

$$(s+3)(s+2)x^{s+1} + (s+2)(s+1)Bx^s = (s+3)(s+2)(-x-c)^{s+1} + (s+2)(s+1)B(-x-c)^s.$$

If $s > 1$ then the only multiple root of left hand side is $x = 0$, and only multiple root of the right hand side is $x = -c$, thus $s = 1$. By comparing degrees in equation (4) we get that $\deg g_2 = 0$, contradiction as $g_2(x)$ has non-zero root. \square

3. A DIOPHANTINE EQUATION

In this section we give sufficient condition on quadrinomials to have only finitely many common integral points. We recall Bilu-Tichy result. To state this theorem we will need the notion of standard pairs over \mathbb{Q} . We list standard pairs of polynomials over \mathbb{Q} in the table

kind	standard pair (or switched)	parameter restrictions
first	$(x^m, ax^r p(x)^m)$	$r < m, \gcd(r, m) = 1, r + \deg p > 0$
second	$(x^2, (ax^2 + b)p(x)^2)$	
third	$(D_m(x, a^n), D_n(x, a^m))$	$\gcd(m, n) = 1$
fourth	$(a^{-m/2} D_m(x, a), -b^{-n/2} D_n(x, b))$	$\gcd(m, n) = 2$
fifth	$((ax^2 - 1)^3, 3x^4 - 4x^3)$	

where a, b are non-zero rationals, m, n are positive integers, r is non-negative integer, $p \in \mathbb{Q}[x]$ is a polynomial (which may be constant) and $D_n(x, a)$ is the n -th Dickson polynomial with parameter a given by the formula

$$D_n(x, a) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-a)^i x^{n-2i}.$$

Now we are ready to recall the theorem of Bilu and Tichy [1]

Theorem 3.1. *Let $f, g \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following assertions are equivalent*

- *The equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator.*
- *We have*

$$f(x) = \varphi(f_1(\lambda(x))), \quad g(x) = \varphi(g_1(\mu(x))),$$

where $\varphi \in \mathbb{Q}[x]$, $\lambda, \mu \in \mathbb{Q}[x]$ are linear polynomials, and (f_1, g_1) is a standard pair over \mathbb{Q} such that the equation $f_1(x) = g_1(y)$ has infinitely many rational solutions with a bounded denominator.

Before stating the main theorem we prove some lemmas concerning decompositions of certain polynomials. As a main ingredient in proofs we will use the classical theorem of Gessel and Viennot [3] concerning determinants with binomial coefficients.

Theorem 3.2. *Let $0 \leq a_1 < a_2 < \dots < a_n$ and $0 \leq b_1 < b_2 < \dots < b_n$ be strictly increasing sequences of non-negative integers. Then the determinant*

$$\det \left(\left[\binom{a_i}{b_j} \right]_{i,j=1,2,\dots,n} \right)$$

is non-negative, and positive iff $b_i \leq a_i$ for all i .

Lemma 3.3. *Let $f, g \in \mathbb{Q}[x]$ be polynomials with rational coefficients, and $u, v \in \mathbb{Q}$ be non-zero rationals such that*

$$f(x) = g(ux + v).$$

Suppose that $g(x)$ has exactly l non-zero terms and $f(x)$ has exactly k non-zero terms, and $n = \deg f = \deg g$. Then the following inequality holds

$$n + 2 \leq k + l.$$

Proof. We put $g(x) = \sum_{i=1}^l C_{n_i} x^{n_i}$, where $(n_i)_{i=1}^l$ is a decreasing sequence of non-zero integers, and C_{n_i} are non-zero rationals. The coefficient at x^j in the polynomial $g(ux+v)$ is equal to

$$u^j v^{-j} \sum_{i=1}^l C_{n_i} \binom{n_i}{j} v^{n_i}.$$

Let us suppose that this coefficient vanish for $j = m_1, m_2, \dots, m_s$, where $s = n + 1 - k \geq l$ and $(m_i)_{i=1}^s$ is a decreasing sequence of non-zero integers. We observe that the vector $(C_{n_1} v^{n_1}, \dots, C_{n_l} v^{n_l})$ is perpendicular to every row of the matrix

$$\left[\binom{n_i}{m_j} \right]_{i=1, \dots, l, j=1, \dots, s}.$$

We put $t = \max\{i | n_i \geq m_i, 1 \leq i \leq l\}$, of course $n = n_1 > m_1$ and so t is well-defined. From Lemma 3.2 we get that the determinant of the matrix

$$M = \left[\binom{n_i}{m_j} \right]_{i=1, \dots, t, j=1, \dots, t}$$

is non-zero. Moreover the vector $(C_{n_1} v^{n_1}, \dots, C_{n_t} v^{n_t})$ is in the kernel of M , therefore $v = 0$. Which contradicts the assumption $v \neq 0$. We proved that $s = n + 1 - k < l$, therefore $n + 2 \leq k + l$. \square

Lemma 3.4. *Let $n_1 > n_2 > \dots > n_s > 0$ be positive integers and A_1, A_2, \dots, A_{s+1} be rational numbers such that $A_1 A_2 \dots A_s \neq 0$, and put $f(x) = A_1 x^{n_1} + \dots + A_s x^{n_s} + A_{s+1}$. If the equation*

$$D_{n_1}(x, \gamma) = f(ux + v)$$

holds for some $u, v, \gamma \in \mathbb{Q}$ such that $u\gamma \neq 0$, then $n_1 \leq 2s$.

Proof. If $v = 0$ then we get

$$D_{n_1}(x, \gamma) = \sum_{i=0}^{\lfloor n_1/2 \rfloor} \frac{n_1}{n_1 - i} \binom{n_1 - i}{i} (-\gamma)^i x^{n_1 - 2i} = f(ux).$$

The polynomial on the left hand side has exactly $\lfloor n_1/2 \rfloor + 1$ non-zero coefficients, while $f(ux)$ has s or $s + 1$ non-zero coefficients. In consequence $\lfloor n_1/2 \rfloor + 1 \leq s + 1$ so $n_1 \leq 2s + 1$. It can't be $n_1 = 2s + 1$ because in that case $A_{s+1} = 0$ and $f(ux)$ has s non-zero coefficients so $\lfloor n_1/2 \rfloor + 1 \leq s$ and thus $n_1 \leq 2s - 1$.

Suppose that $v \neq 0$ and $n_1 \geq 2s + 1$. From Lemma 3.3 we get that

$$\deg D_{n_1}(x, \gamma) = n_1 \leq \lfloor n_1/2 \rfloor + 1 + (s + 1) - 2 = \lfloor n_1/2 \rfloor + s,$$

therefore $n_1 \leq 2s$. \square

Now we are ready to state the main theorems of this section.

Theorem (A). *Let $f(x) = Ax^{n_1} + Bx^{n_2} + Cx^{n_3} + D$, $g(x) = Ex^{m_1} + Fx^{m_2} + Gx^{m_3} + H$ with $f, g \in \mathbb{Q}[x]$, $n_1 > n_2 > n_3 > 0$, $m_1 > m_2 > m_3 > 0$, and $\gcd(n_1, n_2, n_3) = 1$, $\gcd(m_1, m_2, m_3) = 1$, $(m_1, m_2, m_3) \neq (n_1, n_2, n_3)$, $ABC \neq 0$, $EFG \neq 0$ and $n_1, m_1 \geq 9$. Then the equation*

$$f(x) = g(y)$$

has only finitely many integer solutions.

Proof. If the equation $f(x) = g(y)$ has infinitely many integer solutions, then

$$f = \varphi \circ f_1 \circ \lambda, \quad g = \varphi \circ g_1 \circ \mu,$$

where $\varphi, \mu, \lambda, f_1, g_1 \in \mathbb{Q}[x]$ and (f_1, g_1) is a standard pair, μ, λ are linear polynomials.

Let us consider pairs of the first kind. From symmetry we can assume that

$$f_1(x) = x^m, \quad g_1(x) = ax^r p(x)^m$$

for $a \in \mathbb{Q} \setminus \{0\}$, $0 \leq r < m$, $\gcd(r, m) = 1$, $p(x) \in \mathbb{Q}[x]$ and $r + \deg p > 0$. From Theorem 2.5 we get that $\deg \varphi \leq 2$ or $\deg f_1 = \deg g_1 = 1$. If $\deg \varphi = 1$ then we have

$$\varphi^{-1} \circ f(x) = x^m \circ \lambda = (ux + v)^m$$

for some $u, v \in \mathbb{Q}$. The polynomial $\varphi^{-1} \circ f$ has at least three non-zero terms so $v \neq 0$. Therefore $\varphi^{-1} \circ f$ has exactly four non-zero terms and $(ux + v)^m$ has exactly $m + 1$ non-zero terms. Therefore $m = \deg f = 3$, this is a contradiction with $\deg f \geq 9$.

If $\deg \varphi = 2$. Then from Theorem 2.5 we get that $x^m \circ \lambda$ has two or three non-zero terms. As in the previous case we get that $m = 1$ or $m = 2$ therefore $\deg f \leq 4$ this contradicts the assumption $\deg f \geq 9$.

If $\deg f_1 = \deg g_1 = 1$ then $m = 1$, $r = 0$, $\deg p = 1$. So $f = g \circ l$ for some linear l , from Lemma 3.3 we get that either $(m_1, m_2, m_3) = (n_1, n_2, n_3)$ or $\deg f \leq 4 + 4 - 2 = 6$ - a contradiction.

Let us consider pairs of the second kind. From symmetry we can assume that $f_1(x) = x^2$, but then from Theorem 2.5 we get that $\deg(\varphi) \leq 2$ and therefore $\deg f \leq 4$ - a contradiction with $\deg f \geq 9$.

Let us consider pairs of the fifth kind. From symmetry we can assume that $f_1(x) = 3x^4 - 4x^3$, $g_1 = (ax^2 - 1)^3$ from Theorem 2.5 we get that $\deg \varphi \leq 2$. If $\deg \varphi = 1$ then $\deg f = 4$ contradiction with $\deg f \geq 9$. If $\deg \varphi = 2$ then from Theorem 2.5 we get that $(ax^2 - 1)^3 \circ \mu$ has two or three non-zero terms. Let us write $\mu(x) = ux + v$ then we have $(ax^2 - 1)^3 \circ \mu = (au^2x^2 + 2auvx + av^2 - 1)^3$ has non-zero triple root (note that $au^2x^2 + 2auvx + av^2 - 1 = au^2x^2$ implies $av^2 = 1$ and $auv = 0$ contradiction because $u \neq 0$), therefore from Lemma 2.2 we get a contradiction.

Let us consider pairs of the third and fourth kind. In both cases we have $f_1 = aD_m(x, \gamma)$ for some $a, \gamma \in \mathbb{Q} \setminus \{0\}$. From Theorem 2.5 we get that $\deg \varphi \leq 2$. If $\deg \varphi = 1$ then we have $D_m(x, \gamma) = \varphi^{-1} \circ f \circ \lambda^{-1}$ and from Lemma 3.4 we get that $\deg f = m \leq 6$ a contradiction. If $\deg \varphi = 2$ then $f_1 \circ \lambda$ has two or three non-zero terms. Therefore from Lemma 3.4 we get that $\deg f_1 \leq 4$, so $\deg f \leq 8$ - a contradiction. \square

To prove theorem concerning equations with lacunary polynomials we will use the following result of Zannier [12].

Theorem 3.5 (Zannier). *Suppose that $g, h \in \mathbb{C}[x]$ are non-constant polynomials, that $h(x)$ is not of the shape $ax^n + b$ and that $g(h(x))$ has at most $l > 0$ non-zero terms at positive powers. Then $\deg g \leq 2l(l-1)$.*

Now we are ready to prove

Theorem (B). *Let $l \geq 4$ and $n_1 > n_2 > \dots > n_l > 0$, $m_1 > m_2 > m_3 > 0$ be positive integers. Let*

$$f(x) = A_1x^{n_1} + A_2x^{n_2} + \dots + A_lx^{n_l} + A_{l+1} \quad \text{and} \quad g(x) = Ex^{m_1} + Fx^{m_2} + Gx^{m_3}$$

be polynomials with rational coefficients such that $\gcd(n_1, n_2, \dots, n_l) = 1$, $\gcd(m_1, m_2, m_3) = 1$, $A_1A_2 \dots A_l \neq 0$, $EFG \neq 0$ and $m_1 \geq 2l(l-1)$, $n_1 \geq 4$. Then the equation

$$f(x) = g(y)$$

has only finitely many integer solutions.

Proof. If the equation $f(x) = g(y)$ has infinitely many integer solutions, then

$$f = \varphi \circ f_1 \circ \lambda, \quad g = \varphi \circ g_1 \circ \mu,$$

where $\varphi, \mu, \lambda, f_1, g_1 \in \mathbb{Q}[x]$ and (f_1, g_1) is a standard pair, μ, λ are linear polynomials.

Suppose that $\deg \varphi > 2$. From Theorem 2.5 we get that either $g_1 \circ \mu = \rho \circ x^d$ for some linear $\rho \in \mathbb{Q}[x]$, and positive integer $d \geq 2$, or $\deg(g_1 \circ \mu) = 1$. The first case is impossible due to the assumption $\gcd(m_1, m_2, m_3) = 1$. In the second case we get

$$f = \varphi \circ f_1 \circ \lambda = g \circ (g_1 \circ \mu)^{-1} \circ f_1 \circ \lambda.$$

From Zannier's theorem we get that $(g_1 \circ \mu)^{-1} \circ f_1 \circ \lambda = ux^k + v$ for some rationals u, v and positive integer k . We claim that $k = 1$. Suppose that this is not the case, then all non-zero coefficients of f are at powers divisible by k which contradicts the assumption $\gcd(n_1, \dots, n_l) = 1$. In the case of $k = 1$ we have the equation $f = g \circ \rho$ for some linear $\rho \in \mathbb{Q}[x]$. From Lemma 3.3 we get that

$$2l(l-1) \leq \deg g \leq (l+1) + 3 - 2 = l + 2,$$

or $f(x) = g(ux)$ for some non-zero rational u . The former is impossible since $l \geq 4$. The latter is impossible since g has three non-zero terms and f has at least four non-zero terms. We proved that $\deg \varphi \leq 2$.

Let us consider standard pairs of the first kind. Suppose that $g_1(x) = x^m$ for some m . If $\deg \varphi = 1$ then we have

$$\varphi^{-1} \circ g = x^m \circ \mu = (ux + v)^m$$

for some rationals u, v . Suppose that $v = 0$, then $g(x) = \varphi((ux)^m)$ has at most two non-zero terms - a contradiction. In the case of $v \neq 0$ we get that $(ux + v)^m = \varphi^{-1} \circ g$ has $m + 1 = \deg g + 1 > 4$ non-zero terms whereas g has three non-zero terms - a contradiction.

If $\deg \varphi = 2$ then from Theorem 2.5 we get that $g_1 \circ \mu = x^m \circ \mu$ has two or three non-zero terms, as in previous case we get that $m = 1$ or $m = 2$. This is a contradiction with the condition $2m = \deg g > 4$.

Suppose that (f_1, g_1) is a switched pair of the first kind namely $g_1(x) = ax^r p(x)^m, f_1(x) = x^m$. If $\deg \varphi = 1$ then we have

$$a(ux + v)^r p(ux + v)^m = \varphi^{-1} \circ g$$

for some rationals u, v . If $\deg p > 0$ then from Lemma 2.2 we get that $m \leq 3$ which contradicts the fact that $4 < \deg f = \deg(\varphi \circ x^m \circ \lambda) = m$. In the case of $\deg p = 0$ we get that $a(ux + v)^r = \varphi^{-1} \circ g$ and again from Lemma 2.2 we get $r \leq 3$ and thus $\deg g \leq 3$ - a contradiction.

In the case of $\deg \varphi = 2$ we have $\varphi \circ g_1 \circ \mu = g$. We apply Theorem 2.5 to get that $g_1 \circ \mu$ has at most three non-zero terms. We can write

$$g_1 \circ \mu = a(ux + v)^r p(ux + v)^m.$$

If $\deg p > 0$ then from Lemma 2.2 we get that $m \leq 2$. However in this case $\deg f = \deg(\varphi \circ x^m \circ \lambda) \leq 4$ - a contradiction. In the case of $\deg p = 0$ we again apply Lemma 2.2 to get $r \leq 2$. As a consequence $\deg g \leq 4$ - a contradiction.

Observe that (f_1, g_1) cannot be a standard pair of the second kind, since $\deg(\varphi \circ x^2) \leq 4$ and $n_1, m_1 > 4$.

Let us consider pairs of the third and fourth kind. In both cases we have $g_1 = aD_m(x, \gamma)$ for some $a, \gamma \in \mathbb{Q} \setminus \{0\}$. If $\deg \varphi = 1$ then we have $aD_m(x, \gamma) = \varphi^{-1} \circ g \circ \lambda^{-1}$ and from Lemma 3.4 we get that $\deg g = m \leq 6$ a contradiction. If $\deg \varphi = 2$ then $g_1 \circ \lambda$ has two or three non-zero terms. Therefore from Lemma 3.4 we get that $\deg g_1 \leq 4$ so $\deg g \leq 8$ - a contradiction.

Finally let us note that (f_1, g_1) cannot be a standard pair of the fifth kind. Since $\deg g_1 \leq 6$ and $\deg \varphi \leq 2$ we get that $24 \leq 2l(l - 1) \leq \deg g = \deg(\varphi \circ g_1 \circ \mu) \leq 12$ - a contradiction. \square

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